AN ANALYTICAL REAL OPTION REPLACEMENT MODEL WITH DEPRECIATION

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JEL Classifications: D81, G31
Keywords: Asset replacement, two factor model, depreciation tax allowance

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Abstract

A replacement model is presented for a productive asset subject to stochastic input decay, tax allowances due to a deterministic depreciation variable, and a fixed investment cost. The risk neutral valuation function is formulated and optimal trigger levels signalling replacement for the two factors is determined analytically although not as a closed-form solution. We demonstrate that the operating cost trigger level depends on asset age and increases monotonically due to positive volatility changes and that the model solution furnishes the results for certain special cases. The analysis is conducted both for a depreciation schedule specified by the declining balance and straight line method. The comparative analysis shows that although no universal ideal depreciation schedule exists between the two, the declining balance method is preferred. Finally, the solution method is sufficiently tractable to be applied in principle to real option models where time is a critical factor.

1. Introduction

The replacement policy for productive assets is normally governed by the degree of their quality degradation, which is manifested through the stream of possible tax credits attributable to the depreciation schedule in use as well as the operating cash flows that the asset under study generates. For productive assets, especially those belonging to long-lived expensive projects, these depreciation tax allowances distributed over the asset's lifetime can represent significant positive cash flows, which can crucially influence the decision of whether to continue with or to replace the incumbent. By formulating these allowances within a real option model under input decay, an analytical expression for the optimal replacement policy is developed from the risk neutral valuation relationship so that the extent of their significance can be evaluated.

This paper examines the replacement policy for an incumbent productive asset when the stochastic operating cost is described by geometric Brownian motion, tax allowances due to a deterministic depreciation variable are available and when the replacement investment cost is fixed. The after tax risk neutral valuation relationship for the asset value including the embedded replacement option is formulated and analytically determined. The optimal replacement policy is then derived from the economic boundary conditions that yield a set of simultaneous equations from which the optimal trigger levels for the two focal variables are evaluated. Analysis on the stochastic replacement model establishes that solutions to the special cases of an absent depreciation variable and a zero underlying volatility can be derived from the general result. Numerical analysis on the solution reveals real option model paradigm that the trigger level for the stochastic variable and the value relationship are both increasing functions of the underlying volatility. Since the stochastic model is constructed on two alternative depreciation schedules, declining balance and the straight line method, a comparison is conducted on the distinct effect of each schedule on the replacement policy to reveal that although no depreciation schedule is universally ideal, the declining balance method is preferred.

Fundamentally, the applications of real options methods to a decision making context in the presence of uncertainty are founded on the valuation of perpetual American options under risk neutrality, Samuelson (1965), and the deduction that traditional capital budgeting techniques misprice the option value, Myers & Turnball (1976). Amongst the original contributors to real option applications include Tourinho (1979) who values oil shale reserves and determined the oil price trigger level signalling exploitation, McDonald & Siegel (1985) who investigate the abandonment option, McDonald & Siegel (1986) who demonstrate that the optimal investment policy is often to defer in the presence of uncertainty, and Brennan & Schwartz (1985) who from deriving the optimal conditions governing the temporary suspensions of operations and their re-enactment, then proceed to demonstrate the effect of hysteresis.

The first investigations of stochastic replacement models are conducted using a dynamic programming formulation, Bellman (1955), Rust (1987). Subsequent formulations seek to identify the optimal replacement conditions when the asset degradation is described entirely by input decay by ignoring output decay, Feldstein & Rothschild (1974), and the operating cost uncertainty is well described by a known stochastic process. Ye (1990) who treats the behaviour of the operating cost to be arithmetic Brownian motion, demonstrates that the effect of uncertainty is to defer the replacement decision. Similar results are obtained by models grounded on geometric Brownian motion. Mauer & Ott

(1995) devise a sophisticated formulation that is constructed on the after tax risk neutral valuation for a replacement model involving the variations in operating cost, depreciation and salvage price. An additional model is presented by Dobbs (2004). Other real options models related to the productive asset replacement context include Malchow-Møller & Thorsen (2005) on technology replacement, see also Malchow-Møller & Thorsen (2006) and Williams (1997) on real asset redevelopment.

The present model extends the analytical scope of these real option replacement representations by introducing the depreciation schedule as a distinct variable into the formulation based on input decay. The introduction is completed through expressing depreciation as a deterministic time dependent variable. This entails modelling the depreciation variable as geometric Brownian motion with zero underlying volatility for depreciation computed using the declining balance method and as arithmetic Brownian motion with zero underlying volatility when using the straight line method. The incorporation of the additional variable into the formulation means that the valuation function for the asset including its embedded replacement option depends on two distinct factors and that the search for the optimal trigger levels for those two factors requires the analytical solution to a two-dimensional valuation relationship.

Previous multifactor real option models have adopted one of three methods for deriving their results. The first approach pivots on the valuation function possessing the property of homogeneity of degree one. Effectively, this approach treats the phenomenon under study as an exchange option, Margrabe (1978) and Sick (1989), and uses a ratio transformation to reduce the model dimensionality from two to one from which a closed-form solution is generated. Illustrations of this approach include McDonald & Siegel (1986), Williams (1991) and Malchow-Møller & Thorsen (2005). However, since the replacement investment cost is fixed, this approach is not tenable, Adkins & Paxson (2006). The second approach, proposed by Mauer & Ott (1995), conjectures that the depreciation and salvage price variables can be reliably expressed as functions of the operating cost. These substitutions entail the significant compromise that a deterministic depreciation variable can be satisfactorily represented by a stochastic factor and that salvage value is only determined by the operating cost. The third approach employs numerical finite difference methods to solve the multi-factor valuation relationship. Although this approach makes no compromising simplifying assumptions, it does possess disadvantages to the method used in the present formulation. Principally, from the method adopted here, we establish analytically the results for special cases from the general solution. Further, it is possible in principle to derive analytical expressions for various key indicators such as vega.

The outstanding reason for adopting the analytical procedure canvassed in this paper is that its scope of analysis goes beyond the confines of the replacement phenomenon under study. Since the valuation function depends on two factors, the solution method is applicable to other two factor models for which the property of homogeneity of degree one cannot be invoked for sound logical reasons. Secondly, the introduction of a time dependent variable into the formulation and the analytical derivation of the resulting valuation relationship mean that real option formulations involving a time dependent variable should in principle be amenable to analysis and yield a quasi-analytical solution. Potentially, this paves the way for developing and solving real option models in which time is a critical factor such as the replacement of assets which have a finite life.

This paper is organised in the following way. In section 2, we formulate and develop the analytical solution to the stochastic replacement real option model for an asset whose depreciation follows a declining balance process. By modifying the parametric values, it is demonstrated that expressions for the optimal trigger level for the operating cost is derivable from this general model. In the following section, we conduct a variety of simulation experiments to reveal the behaviour of the solution and to supply a greater insight into the nature of the model. Section 4 re-examines the stochastic replacement real option model for a straight line depreciation charge and an investigation of its sensitivity to parametric changes is performed in the following section. A comparison of the model results under the declining balance and straight line method is discussed in section 6. The conclusion in section 7 brings the paper to a close. The deterministic replacement model for the two variant forms of depreciation represents a benchmark for assessing the model results and this analysis is relegated to Appendix A.

2. Replacement Opportunity with Declining Depreciation

Consider a capital asset deployed in a productive process, which has a significant bearing on business performance. This asset suffers degradation in quality due to usage and its degree of deterioration is reflected through increases in its operating cost. As the operating costs for this asset become increasingly more inferior relative to those of a newly installed replica, a decision has to be reached on whether to continue with the incumbent asset or to replace it with a replica having a superior operating performance. Under uncertainty, the solution for the asset replacement model is determined from optimising the expected present value of the after tax uncertain stream of net cash flows attributable to the asset for all possible replacement policies. The solution for the model with a single stochastic variable is characterised by an upper critical limit, beyond which replacement is the prescribed policy. Introducing a depreciation charge into the model, even though it is a deterministic variable, alters the critical limit from a single point level to a two-dimensional discriminatory boundary. The optimal policy for the replacement model involving a stochastic operating cost variable and a deterministic depreciation charge variable is jointly settled by their prevailing values. The discriminatory boundary, which separates the region of continuance from replacement, is evaluated by comparing the expected present value for the incumbent asset with that for a replica with its improved performance less the fixed investment cost incurred from obtaining the improvement net of any residual depreciation tax shield.

For some point of time, the operating cost for the asset under consideration is denoted by the time dependent stochastic variable C. The notation we use in the stochastic replacement model ignores the time subscript since its omission leads to no confusion. In their real options analysis of capital replacement, Mauer & Ott (1995) and Dobbs (2004) assume that the stochastic cost behaviour is adequately represented by a geometric

Brownian motion process with positive drift. Similarly, we will adopt the same process by specifying the before tax operating cost as:

$$dC = \alpha_C C dt + \sigma_C C dz_C, \tag{1}$$

where α_C is its instantaneous drift rate, σ_C is the instantaneous volatility rate, and dz_C is the increment of a standard Wiener process. The operating cost is a measure of input decay and since the asset deteriorates with age, α_C is expected to be positive.

Since the capital allowances attributed to the asset under consideration acts as a tax shield, this factor influences the replacement decision and plays a role in determining the discriminatory boundary. The capital allowance is represented by the depreciation charge D, which is calculated on the basis of a declining balance and is described by a deterministic geometric process:

$$dD = -\alpha_D D dt, \tag{2}$$

where α_D is the constant proportional depreciation rate. Since the depreciation charge is described deterministically by a time dependent variable then from knowing the depreciation charge at the time of replacement, the prevailing depreciation charge level determines the time elapsed since the last replacement.

The degradation the asset suffers is assumed to be due to input decay and impairments in performance arising from usage are manifested in its operating cost. Output decay is treated as not relevant for the model context and the revenues generated by the asset under consideration remain at the constant level P_0. At the replacement event, replacing the incumbent by a replica asset incurs a fixed known investment cost, which is denoted by K. The replacement investment is considered to be irreversible and the asset owner is unable to recoup any of the capital outlay on its discharge. Any salvageable value available on discharge is assumed to be constant and is absorbed by the replacement investment cost. If the replacement investment cost carries any instantaneous tax credits, these are fully absorbed by K. When the incumbent is replaced by a superior replica asset, the operating cost is restored to the superior original level C_0 and the depreciation charge level becomes D_0. If the investment cost is fully depreciable for tax purposes, D_0

and K are related by $D_0 = \theta K$. Although this adjustment can be accommodated in the initial formulation, we leave it open since it is straightforward to make the refinement by modifying the model solution.

The possession of an operating asset conveys to its owner a portfolio of options including the option to replace. Although other operating opportunities such as changes in scale or temporary suspension may be available, we assume that the replacement decision for the asset under consideration is made in isolation to any other enacted policies and that these other flexibilities are absent. We introduce the valuation function F, which is defined as the value of the incumbent asset including its embedded replacement option. This valuation function depends on the critical variables that influence the replacement policy. These are the operating cost for the incumbent asset and its depreciation charge, and $F = F(C,D)$. The value of the asset in use is determined in part by its attributed after tax cash flows:

$$(P - C)(1 - \tau) + D\tau .$$

where τ denotes the relevant corporate tax rate. By assuming complete markets, standard contingent claims analysis can be applied to the asset with value F to determine its risk neutral valuation relationship as a partial differential equation (the derivation is presented in Appendix B), Constantinides (1978), Mason & Merton (1985). The valuation relationship is:

$$\tfrac{1}{2}\sigma_C^2 C^2 \frac{\partial^2 F}{\partial C^2} + \theta_C C \frac{\partial F}{\partial C} - \theta_D D \frac{\partial F}{\partial D} + (P - C)(1 - \tau) + D\tau - rF = 0 , \tag{3}$$

where r denotes the risk-free rate of interest, θ_C the risk-adjusted drift rate for the operating cost and $\theta_D = \alpha_D$. An alternative derivation for (3) relies on using an arbitrage argument, Shimko (1992), in which case $r = \mu$ and $\theta_C = \alpha_C$.

The nature of F can be partly resolved by examining its behaviour as the variables approach their limiting values. Ignoring higher derivatives greater than one, the particular solution F_P to (3) is:

$$F_P = \frac{P_0(1-\tau)}{r} - \frac{C(1-\tau)}{r-\theta_C} + \frac{D\tau}{r+\theta_D}.$$

F, which has to be non-negative otherwise there would be no initial asset investment, is conceived as the combination of the incumbent asset value F_V and the replacement option F_R, with $F = F_V + F_R$. Since the option value is always non-negative, then $F \geq F_V$, Trigeorgis (1996). Assuming an infinite lifetime:

$$F_V(t \to \infty) = \frac{P_0(1-\tau)}{r} - \frac{C(1-\tau)}{r-\theta_C} + \frac{D\tau}{r+\theta_D} = F_P.$$

When the operating costs for the incumbent asset become increasingly adverse and approach infinity, there would normally exist a cogent economic justification for replacing the incumbent. For $C \to \infty$, the asset value becomes negative and is dominated by the replacement option value which tends to infinity. In contrast, when the operating costs become increasingly favourable and approach zero, no economic justification exists for replacing the incumbent. For $C \to 0$, the asset value is strongly positive but the replacement option value is close to zero. We now consider the effect of the limiting values for the depreciation charge on the replacement option. Seemingly, we may wish to contend that old assets are probably inefficient and ready for replacement, and so the replacement option value is greatest when the depreciation charge tends to zero. However, that is not the case because of the effect of the residual depreciation charge on the replacement investment cost. Since the prevailing depreciation charge directly and positively influences the residual depreciation tax shield, which in turn lowers the replacement investment cost, the prevailing depreciation charge exerts its greatest pressure on reducing the replacement investment cost when its value is at its maximum level. This effect is palpable from the value matching relationship (8) since any reduction in the replacement investment cost caused by the residual depreciation shield $\frac{\tau D}{\theta_D}$ is always greater than the present value of the depreciation tax shield in the limit $\frac{\tau D}{r+\theta_D}$.

The option value for replacing the incumbent is positively influenced by the prevailing depreciation charge and it attains its greatest value when the depreciation charge at its maximum and its lowest value when it is at its minimum. Collectively, we can describe these limiting boundary conditions by:

$$F_R\left(C \to 0, D\right) \to 0, F_R\left(C \to \infty, D\right) \to \infty, F_R\left(C, D \to 0\right) \to 0, F_R\left(C, D \to \infty\right) \to \infty . \tag{4}$$

The simplest kind of function satisfying (3) takes the generic form:

$$F = AD^{\beta}C^{\eta} + \frac{P_0\left(1-\tau\right)}{r} - \frac{C\left(1-\tau\right)}{r-\theta_C} + \frac{D\tau}{r+\theta_D}, \tag{5}$$

where A denotes a generic coefficient whose value is to be determined. This generic form can be justified in two ways. Ignoring the after tax cash flow element, (3) is similar to the valuation relationship formulated by McDonald & Siegel (1986) in their analysis of an investment option. They express the valuation function as the solution to a two-variable partial differential equation as a product power function. Although the partial differential valuation relationships for the two models are not identical, (3) and theirs are exactly the same when the variance of one variable is set to equal zero, so the solution to their relationship with a zero variance for the relevant variable is the solution to (5). McDonald & Siegel (1986) require that the product power function exhibits homogeneity of degree one. We do not impose this condition on (5) and the sum of the parameters $\beta+\eta$ is permitted to be free. Second, (5) is the solution to (3) for the following characteristic equation:

$$Q\left(\beta, \eta\right) = \tfrac{1}{2}\sigma_C^2\eta\left(\eta-1\right) + \theta_C\eta - \theta_D\beta - r = 0 . \tag{6}$$

This is the bivariate equivalent to the characteristic equation formulated for a single variable model, Dixit & Pindyck (1994). Unlike the single variable case, additional information required before the solution values for β and η can be determined. The solutions for β and η are found from the boundary conditions. Their values are identified by the point of intersection for the function Q and the function distilled from the value matching relationship and associated smooth pasting condition. Since the function Q specifies a parabola that exerts a presence in all four quadrants, the solution values for β and η may possibly belong to any of the four quadrants, that is:

$$\begin{aligned} \mathrm{I}&:\{\beta_1,\eta_1\}\ \beta_1 \geq 0, \eta_1 \geq 0;\\ \mathrm{II}&:\{\beta_2,\eta_2\}\ \beta_2 \geq 0, \eta_2 \leq 0;\\ \mathrm{III}&:\{\beta_3,\eta_3\}\ \beta_3 \leq 0, \eta_3 \geq 0;\\ \mathrm{IV}&:\{\beta_4,\eta_4\}\ \beta_4 \leq 0, \eta_4 \leq 0. \end{aligned}$$

This suggests that the specific form of (5) is:

$$F = \sum_{i=1}^{4} A_i D^{\beta_i} C^{\eta_i} + \frac{P_0(1-\tau)}{r} - \frac{C(1-\tau)}{r-\theta_C} + \frac{D\tau}{r+\theta_D}.$$

By invoking the limiting boundary conditions (4), $A_2 = A_3 = A_4 = 0$ and the specific valuation function simplifies to:

$$F = A_1 D^{\beta_1} C^{\eta_1} + \frac{P_0(1-\tau)}{r} - \frac{C(1-\tau)}{r-\theta_C} + \frac{D\tau}{r+\theta_D}. \tag{7}$$

The switch between assets occurs when the value of the incumbent asset attains the value of the replica less the net replacement investment cost, where the value is determined from the combined expected after tax net cash flow and the value of the replacement opportunity and is specified by (7). At the replacement event, the model variables C and D simultaneously achieve their respective trigger levels $\hat{C}$ and $\hat{D}$. Unlike the single variable real option models where the trigger level is represented by a single point, the composite trigger level for the two variable model under consideration is described by an uncountable set of paired trigger levels $\{\hat{C}, \hat{D}\}$. There exists for this two variable model any number of distinct trigger level possibilities because of the trade-off that exists between the operating cost and depreciation trigger levels. If replacement is economically viable for a specific pair of trigger levels, a small change in one trigger level accommodated by a commensurate change in the other also satisfies the condition conducive to optimal replacement. The uncountable set of paired trigger levels is represented by the function G with $G(\hat{C}, \hat{D}) = 0$.

At the replacement event, the difference in the values for the incumbent and replica assets has to equal the net replacement investment cost. The depreciation for the incumbent asset at replacement is $\hat{D}$ and its residual depreciation defined as the accumulation over its remaining life is $\hat{D} / \theta_D$. By assuming that the whole residual depreciation is allowable against tax, the residual depreciation tax credit is $\tau\hat{D} / \theta_D$ and the net replacement investment cost is $K - \tau\hat{D} / \theta_D$. Similar to Mauer & Ott (1995) and Dobbs (2004), the economic condition signaling replacement is:

$$F\left(\hat{C}, \hat{D}\right) = F\left(C_0, D_0\right) - K + \tau\hat{D} / \theta_D$$

which can be expressed as:

$$A_1\hat{D}^{\beta_1}\hat{C}^{\eta_1} - \frac{\hat{C}\left(1-\tau\right)}{r-\theta_C} + \frac{\hat{D}\tau}{r+\theta_D} = A_1 D_0^{\beta_1} C_0^{\eta_1} - \frac{C_0\left(1-\tau\right)}{r-\theta_C} + \frac{D_0\tau}{r+\theta_D} - K + \frac{\tau\hat{D}}{\theta_D}. \tag{8}$$

Associated with the value matching relationship (8) are two smooth pasting conditions with respect to C and D. From these, we establish that:

$$A_1\hat{D}^{\beta_1}\hat{C}^{\eta_1} = \frac{\left(1-\tau\right)\hat{C}}{\eta_1\left(r-\theta_C\right)} = \frac{\tau r\hat{D}}{\beta_1\theta_D\left(r+\theta_D\right)} \geq 0 \tag{9}$$

Since the option value is non-negative, $A_1 \geq 0$, which corroborates that both β_1 and η_1 are non-negative. By using (9), we eliminate A_1 from (8) to yield:

$$\frac{\hat{C}\left(1-\tau\right)}{\eta_1\left(r-\theta_C\right)}\left(\eta_1 + \beta_1 - 1 + \left(\frac{D_0}{\hat{D}}\right)^{\beta_1}\left(\frac{C_0}{\hat{C}}\right)^{\eta_1}\right) = \frac{C_0\left(1-\tau\right)}{\left(r-\theta_C\right)} - \frac{D_0\tau}{\left(r+\theta_D\right)} + K. \tag{10}$$

Since both sides of (8) have the same sign, $\beta_1 + \eta_1 > 1$.

The reduced forms of the value matching relationship and smooth pasting condition, (10) and (9) respectively, and the characteristic equation (6) collectively constitute the model for the value of an active productive process that embodies a replacement option to exchange the incumbent asset with a replica. These three equations are sufficient to determine the discriminatory boundary separating the continuance from the replacement region. Although this model comprises four unknowns in total, $\hat{C}$, $\hat{D}$, β_1 and η_1, the requirement for model determinacy is satisfied since the construction of the discriminatory boundary requires one of the variables to have a pre-specified value and the function $G(\hat{C},\hat{D})=0$ makes up the missing equation. No closed-form analytical solution exists for the general model and we have to resort to determining the model solution by solving numerically the set of simultaneous equations. In the next section, we present the simulation analysis and discuss the results that it generates.

An alternative interpretation of the reduced form value matching relationship is from recognising that depreciation is only a time dependent variable. From (2) $\hat{D}=D_0 e^{-\theta_D \hat{T}}$ where $\hat{T}$ denotes the elapsed time between replacements. This means that the value matching relationship can be cast in terms of the optimal elapsed time. By replacing $\hat{D}$ by $\hat{T}$ and eliminating $\hat{C}$, (10) becomes:

$$\frac{\tau r D_0 e^{-\theta_D \hat{T}}}{\beta_1 \theta_D (r+\theta_D)}\left(\eta_1+\beta_1-1+e^{\beta_1 \theta_D \hat{T}} C_V^{\eta_1}\right)=\frac{C_0(1-\tau)}{(r-\theta_C)}-\frac{D_0 \tau}{(r+\theta_D)}+K, \tag{11}$$

where:

$$C_V=\frac{(1-\tau)C_0}{\tau D_0 e^{-\theta_D \hat{T}}}\frac{\beta_1 \theta_D (r+\theta_D)}{\eta_1 r (r-\theta_C)}.$$

The revised value matching relationship (11) and the characteristic equation (6) now form the replacement model and by setting $\hat{T}$ to equal a pre-specified value, solutions to the unknown parameters β_1 and η_1 can be determined from these two simultaneous equations. From these values, $\hat{C}$ can be found from (9). This means that the operating cost trigger level is time dependent and the optimal replacement policy depends on the age of the incumbent asset. Older assets are retired and replaced by a replica at a different operating cost trigger level than younger assets and asset usage plays a significant role in governing the replacement policy.

By formulating the evolution of both the operating cost and depreciation, the stochastic replacement model adopts a general form from which certain special cases can be derived. The deterministic replacement model emerges when the volatility of the operating cost is set to equal zero. In Appendix C, we establish that when the stochastic model is constructed from applying an arbitrage approach grounded on dynamic programming, the stochastic formulation for a zero operating cost volatility simplifies to yield the solution to the deterministic replacement model. Second, the model presented by Dobbs (2004) that excludes the depreciation variable from the formulation is a special case of the general replacement model when adjustments are made to the depreciation variable. The omission of the depreciation variable from the general replacement model implies that θ_D is set to equal zero and the variable D is excluded from the valuation relationship, so $\beta_1 = 0$. It follows that (10) simplifies to the solution supplied by Dobbs (2004):

$$\frac{\hat{C}(1-\tau)}{\eta_1(r-\theta_C)}\left(\eta_1 - 1 + \left(\frac{C_0}{\hat{C}}\right)^{\eta_1}\right) = \frac{C_0(1-\tau)}{(r-\theta_C)} + K, \tag{12}$$

with:

$$\eta_1 = \left(\tfrac{1}{2} - \frac{\theta_C}{\sigma_C^2}\right) + \sqrt{\left(\tfrac{1}{2} - \frac{\theta_C}{\sigma_C^2}\right)^2 + \frac{2r}{\sigma_C^2}}.$$

A similar form to (12) is generated by modifying the definition of the residual depreciation tax credit. By treating the residual depreciation as the discounted depreciation accumulated over its remaining lifetime, the residual depreciation equals $D/(\mu+\theta)$ and its tax credit becomes $\tau D/(\mu+\theta)$. When this value is substituted in (8), the depreciation trigger level $\hat{D}$ appears in the resulting value matching relationship only through the option element and consequently, its exponent β_1 is zero. The solution to the replacement model that has a residual depreciation specified by discounting is:

$$\frac{\hat{C}(1-\tau)}{\eta_1(r-\theta_C)}\left(\eta_1 - 1 + \left(\frac{C_0}{\hat{C}}\right)^{\eta_1}\right) = \frac{C_0(1-\tau)}{(r-\theta_C)} - \frac{D_0\tau}{r+\theta_D} + K, \tag{13}$$

with η_1 defined as above.

Although no closed-form solution exists for the three replacement model variants, it is possible to discern the comparative extent of the solutions they produce. Under the Dobbs (2004) model as specified by (12), there is no residual depreciation tax credit to reduce the replacement investment cost and its operating cost trigger level is the greatest of the three since the productive asset has to operate the longest to render sufficient compensation to balance the greatest investment cost. When the residual depreciation is measured by its present value, the residual depreciation tax credit reduces the replacement investment cost and consequently, the operating cost trigger level for this model as specified by (13) is less than the level supplied by the Dobbs (2004) model. Finally, the residual depreciation tax credit attains its greatest value for the residual depreciation that is measured by its accumulation over its remaining potential life. This measure consequently reduces the effective investment cost by the greatest amount and the operating cost trigger level for this model as indicated its value matching relationship (10) is the least amongst the three variants.

3. Simulation Results for Declining Balance Model

Further insights into the nature of the replacement model founded on the stochastic operating cost and deterministic depreciation variable is leveraged through numerical simulations and sensitivity analysis. Because no closed-form solution exists for the model variants, a comparative evaluation of their properties and the identification of any shortcomings is only achievable through the use of numerical techniques. Our primary aim is to compare the trigger levels yielded by the models proposed by Mauer & Ott (1995) and Dobbs (2004) relative to those produced by the present formulation. We then proceed to penetrate the behaviour of the present formulation by examining the way the solution changes due to variations in key variables. The numerical analysis is conducted using the base case data that is exhibited in Table 1. This ignores the value for the constant revenue level P_0 since it is not included in the solutions. In the base case, the initial depreciation level D_0 for a replica is purposely set to equal the replacement investment cost K adjusted by the declining balance rate θ_D with $D_0 = \theta_D K$ so that variations in these factors naturally percolate through into the computed trigger levels.

Table 1
Base Case Data

Replacement investment cost	K	100
Initial operating cost for a replica	C_0	10
Risk neutral operating cost drift rate	θ_C	4%
Operating cost volatility	σ_C	25%
Initial depreciation charge	D_0	10
Depreciation declining balance rate	θ_D	10%
Risk-free interest rate	r	7%
Relevant corporate tax rate	τ	30%

The discriminatory boundaries as mapped out by the respective trigger level for the models proposed by Mauer & Ott (1995) and Dobbs (2004) and for the present formulation are collectively presented in Figure 1. This reveals that the discriminatory boundary for the present formulation, which is represented by the line AB, is a declining relationship between the respective trigger levels. The operating cost trigger level is at its lowest value when the replica is just installed for $\hat{D} = 10$. As the depreciation trigger level decreases and the incumbent asset grows old, the operating cost trigger level increases until it attains its maximum for $\hat{D} = 0$ when its age reaches infinity. The discriminatory boundary AB distinguishes the regions of continuance and replacement. The appropriate policy is to replace the incumbent when its prevailing operating cost and depreciation values are located above the line AB or to continue operations with the incumbent when otherwise.

The discriminatory boundaries for the Mauer & Ott (1995) and Dobbs (2004) models are represented by horizontal straight lines since depreciation is respectively related to the cost trigger level or it is ignored. For both models, the replacement region lies above its discriminatory boundary. We know from section 2 that the discriminatory boundary for the Dobbs (2004) model has to be situated above that for the present formulation. The location of the Mauer & Ott (1995) discriminatory boundary relative to that for the present formulation is obscured since it depends on the base case values. Changes in the base case values may sufficiently lower the Mauer & Ott (1995) discriminatory boundary to enable it to intersect the line AB. Notwithstanding, the model proposed by Mauer & Ott (1995) is founded on the compromise that the deterministic depreciation variable can be reliably represented by a function of the stochastic operating cost variable. This model produces a discriminatory boundary that is horizontal instead of one characterised by a declining relationship and this difference is likely to be more pronounced for younger rather than older assets.

The profiles of β_1 and η_1 due to variations in the depreciation trigger level for the two variable replacement model are presented in Figure 2. This figure reveals that the values of both these parameters are not fixed, unlike the case for an effectively single variable real option model, but change with the value of the depreciation trigger level. Over the relevant range, the sum of the two parameters always exceeds one. They are increasing functions of $\hat{D}$, and attain their maximum levels at $\hat{D} = 10$ for a just installed asset and their minimum levels at $\hat{D} = 0$ for an infinitely aged asset. When the asset age is infinite, $\beta_1 = 0$ and η_1 takes on the value prescribed by the replacement model for a residual depreciation evaluated according to its present value, (13).

4. Replacement Opportunity with Straight Line Depreciation

Since the developments of the stochastic replacement models for depreciation measured according to the declining balance and straight line methods follow an identical structure, we present in this section only those aspects of the analysis that are dissimilar. The method for solving the straight line depreciation replacement model involves maximizing the expected present value of their after tax net cash flows for all possible replacement policies. The valuation relationship is based on risk neutrality, see Appendix B. The solutions are described by a discriminatory boundary that separates the continuance and replacement regions and which is found from comparing the net expected present values for the incumbent asset and the replica. The fundamental difference between the two models lies in the specification of the depreciation, and this has consequences for the analysis. The remaining variables maintain their definitions from section 2.

The quality degradation for the asset under consideration is due to input decay and is manifested by an operating cost evolution described by (1). Unlike declining balance depreciation whose value does not reach zero in a finite time, straight line depreciation introduces a complication into the formulation because of its asymmetric behaviour due to its depletion within a finite time. When the depreciation balance is exhausted, the depreciation level falls to zero and remains at zero until the asset is replaced. On the installation of the replica, its cumulative depreciation D_C over a notional lifetime N is allocated evenly over its lifetime with periodic charges $D_N = D_C / N$. The remaining cumulative depreciation charge for the asset with elapsed lifetime t is X_t so $X_0 = D_C$ and $X_{t \geq N} = 0$. Straight line depreciation entails that the remaining cumulative depreciation charge declines at the constant absolute rate D_N for $t \leq N$:

$$dX = -D_N dt. \tag{14}$$

The value for the incumbent asset including its replacement option for straight line depreciation is denoted by F_2, which depends on the operating costs C and the remaining cumulative depreciation for $X \geq 0$ or $t \leq N$, and from there on by only its operating cost:

$$F_2 = \begin{cases} F_{21}(C, X) \text{ for } X > 0, \\ F_{22}(C) \text{ for } X = 0. \end{cases} \tag{15}$$

When a zero cumulative depreciation charge is attained, the asset shares identical values under the two regimes with $F_{21}(C,\ 0) = F_{22}(C)$. Identification of the optimal replacement policy for any time t requires that we first examine F_{22} and then progress to consider F_{21} since its derivation depends on F_{22}.

For $t \geq N$ when the remaining cumulative depreciation is zero, the straight line depreciation replacement model reduces to the original model in the absence of depreciation and reverts to the formulation as proposed by Dobbs (2004). This means that the valuation function F_{22} for the model with straight line depreciation is:

$$F_{22} = B_2 C^{\eta_1} + \frac{P_0(1-\tau)}{r} - \frac{C(1-\tau)}{r-\theta_C}.$$

and its discriminatory boundary is specified by $\hat{C}$ in (12).

For $t < N$ when the remaining cumulative depreciation is positive, the risk neutral valuation relationship for F_{21} is:

$$\tfrac{1}{2}\sigma_C^2 C^2 \frac{\partial^2 F_{21}}{\partial C^2} + \theta_C C \frac{\partial F_{21}}{\partial C} - D_N \frac{\partial F_{21}}{\partial X} - rF_{21} + (P_0 - C)(1-\tau) + \tau D_N = 0. \tag{16}$$

The solution to (16) is identified by splitting the partial differential equation into its homogenous element that reflects the replacement option value and the particular element that governs the long run asset value. The homogenous element is:

$$\tfrac{1}{2}\sigma_C^2 C^2 \frac{\partial^2 F_{21}}{\partial C^2} + \theta_C C \frac{\partial F_{21}}{\partial C} - D_N \frac{\partial F_{21}}{\partial X} - rF_{21} = 0,$$

whose generic solution takes the form:

$$F_{21} = B_1 C^{\psi} e^{\lambda X}.$$

As before, we adopt a product function as the solution to the homogenous element except that one of its components is specified by $e^{\lambda X}$ since the partial differential term with respect to X does not involve X as a coefficient. Single variable real option models grounded on arithmetic Brownian motion are discussed by Shimko (1992). By substituting the solution in the homogenous element, we demonstrate that the homogenous element is satisfied with characteristic equation:

$$Q_2(\psi,\lambda) = \tfrac{1}{2}\sigma_C^2 \psi(\psi-1) + \theta_C \psi - D_N \lambda - r = 0. \tag{17}$$

By adopting a similar argument as described in section 2 that explains the signs of the exponents, we conclude that both ψ and λ are non-negative. It follows that:

$$F_{21} = B_{11} C^{\psi_1} e^{\lambda_1 X}. \tag{18}$$

The particular element of (16) is:

$$\theta_C C \frac{\partial F_{21}}{\partial C} - D_N \frac{\partial F_{21}}{\partial X} - rF_{21} + (P_0 - C)(1-\tau) + \tau D_N = 0, \tag{19}$$

whose solution is:

$$F_{21} = \frac{P_0\left(1-\tau\right)}{r} - \frac{C\left(1-\tau\right)}{r-\theta_C} + \frac{D_N\tau}{r} + ae^{bX}$$

The unknown parameter b is found by substituting F_{21} in (19) to reveal that $b = -r / D_N$. The complete solution to (16) is derived by stitching together the homogenous (18) and particular elements to yield:

$$F_{21} = B_{11}C^{\psi_1}e^{\lambda_1 X} + \frac{P_0\left(1-\tau\right)}{r} - \frac{C\left(1-\tau\right)}{r-\theta_C} + \frac{D_N\tau}{r} + ae^{-rX/D_N}$$

When the remaining cumulative depreciation charge is zero, $F_{21}\left(C,\ 0\right) = F_{22}\left(C\right)$. This implies that $a = -\tau D_N / r$ Also when $X = 0$, $B_{11} = B_2$, $\psi_1 = \eta_1$ and $\lambda_1 = 0$. It follows that:

$$F_{21} = B_{11}C^{\psi_1}e^{\lambda_1 X} + \frac{P_0\left(1-\tau\right)}{r} - \frac{C\left(1-\tau\right)}{r-\theta_C} + \frac{D_N\tau\left(1-e^{-rX/D_N}\right)}{r}. \tag{20}$$

The quantity $1-e^{-\frac{rX}{D_N}}$ is interpreted as a finite lifetime adjustment term, which equals zero for $X = D_N$ where X / D_N denotes the fraction of the remaining time before the depreciation is fully depleted. Note that $D_N\tau\left(1-e^{-rX/D_N}\right)/r$ represents the annuity discounted at the risk-free rate r with a lifetime X / D_N. The fundamental distinction between the two replacement models is that the straight line variant has a finite lifetime adjustment term to account for the eventual depletion of depreciation and the form of the product function.

For $\hat{X} > 0$, the remaining cumulative depreciation is positive. At the replacement event, the difference between the values for the replica and the incumbent asset has to equal the net replacement investment cost, where the value is determined collectively from the net benefits. Since the remaining cumulative depreciation is denoted by $\hat{X}$ at the replacement event and assuming that the whole amount is allowable for tax purposes, the residual depreciation tax credit for the straight line method is $\tau\hat{X}$ and the net replacement investment cost becomes $K - \tau\hat{X}$. The value matching relationship given by $F_{21}\left(\hat{C},\hat{X}\right) = F_{21}\left(C_0, X_0\right) + \tau\hat{X} - K$ is expressed as:

$$
\begin{aligned}
&B_{11}\hat{C}^{\psi_1}e^{\lambda_1\hat{X}} - \frac{\hat{C}(1-\tau)}{r-\theta_C} + \frac{D_N\tau\left(1-e^{-r\hat{X}/D_N}\right)}{r} \\
&\qquad = B_{11}C_0^{\psi_1}e^{\lambda_1 X_0} - \frac{C_0(1-\tau)}{r-\theta_C} + \frac{D_N\tau\left(1-e^{-rN}\right)}{r} - K + \tau\hat{X}.
\end{aligned} \tag{21}
$$

The smooth pasting conditions associated with (21) is represented by:

$$
B_{11}\hat{C}^{\psi_1}e^{\lambda_1\hat{X}} = \frac{\hat{C}(1-\tau)}{\psi_1(r-\theta_C)} = \frac{\tau\left(1-e^{-r\hat{X}/D_N}\right)}{\lambda_1}. \tag{22}
$$

Since the replacement option element $B_{11}\hat{C}^{\psi_1}e^{\lambda_1\hat{X}}$ always takes on a non-negative value, then ψ_1 and λ_1 are both non-negative. This corroborates our earlier conjecture. Further, a positive change in the remaining cumulative depreciation charge produces an increase in the replacement option value because of the presence of the residual depreciation tax shield in the value matching condition. Using (22) to eliminate B_{11} from (21) yields:

$$
\begin{aligned}
&\frac{\hat{C}(1-\tau)}{\psi_1(r-\theta_C)}\left(\psi_1 - 1 + \frac{C_0^{\psi_1}e^{\lambda_1 X_0}}{\hat{C}^{\psi_1}e^{\lambda_1\hat{X}}}\right) - \frac{D_N\tau\left(1-e^{-r\hat{X}/D_N}\right)}{r} \\
&\qquad = \frac{C_0(1-\tau)}{r-\theta_C} - \frac{D_N\tau\left(1-e^{-rN}\right)}{r} + K - \tau\hat{X}.
\end{aligned} \tag{23}
$$

The reduced forms of the value matching relationship and smooth pasting conditions, (23) and (22) respectively, and the characteristic equation (17) collectively constitute the valuation model for an active productive process embodying a replacement option when depreciation is measured according to the straight line method for a positive cumulative residual depreciation. Although the model is composed of three equations and contains the four unknowns $\hat{C}$, $\hat{X}$, ψ_1, and λ_1, the model is sufficient because of the presence of the discriminatory boundary $G_2\left(\hat{C},\hat{X}\right)=0$. No closed-form analytical solution exists for this model and it is necessary to recourse to numerical methods to generate the solution. The model can be expressed in terms of $\hat{T}$, which denotes the time elapsed between successive replacements, by substituting $\hat{X} = D_C - \hat{T}D_N$ in (22) and (23).

For $\hat{X} = 0$ when the remaining cumulative depreciation is positive, the model has to be modified to accommodate the absence of the residual depreciation tax credit and that the installation of the replica confers a depreciation tax shield over its lifetime. At the replacement event the value matching relationship, which is determined from $F_{22}(\hat{C}) = F_{21}(C_0, X_0) - K$ where $\lambda_1 = 0$ and $B_{11} = B_2$ since $X = 0$, and is expressed as:

$$B_2\hat{C}^{\psi_{12}} - \frac{\hat{C}(1-\tau)}{r-\theta_C} = B_2 C_0^{\psi_{12}} - \frac{C_0(1-\tau)}{r-\theta_C} + \frac{D_N \tau\left(1-e^{-rN}\right)}{r} - K, \tag{24}$$

with:

$$\psi_{12} = \left(\tfrac{1}{2} - \frac{\theta_C}{\sigma_C^2}\right) + \sqrt{\left(\tfrac{1}{2} - \frac{\theta_C}{\sigma_C^2}\right)^2 + \frac{2r}{\sigma_C^2}} \geq 1.$$

The associated smooth pasting condition is expressed:

$$B_2\hat{C}^{\psi_{12}} = \frac{\hat{C}(1-\tau)}{\psi_{12}(r-\theta_C)},$$

which demonstrates that $\psi_{12} \geq 0$ for the replacement option element to be non-negative. Substituting the smooth pasting condition in (24) yields:

$$\frac{\hat{C}(1-\tau)}{\psi_{12}(r-\theta_C)}\left(\psi_{12} - 1 + \frac{C_0^{\psi_{12}}}{\hat{C}^{\psi_{12}}}\right) = +\frac{C_0(1-\tau)}{r-\theta_C} - \frac{D_N \tau\left(1-e^{-rN}\right)}{r} + K. \tag{25}$$

Because the residual depreciation is depleted and $\hat{X} = 0$, (25) supplies a single solution value for $\hat{C}$. This result differs from the solution proposed by Dobbs (2004) (12) by the inclusion of the term representing the present value of the depreciation tax credit. It is straightforward to demonstrate that the value matching relationships for this model when $\hat{X} > 0$ and $\hat{X} = 0$, (21) and (24) respectively, are identical for $\hat{X} = 0$.

Finally, it is possible to demonstrate that stochastic model is general from which the deterministic replacement solution can be derived for a zero operating cost volatility; the proof is provided in Appendix C.

5. Simulation Results for Straight Line Method

We examine the effect of parametric changes on the solution results through the application of sensitivity analysis in order to gain an insight into its behaviour. Specifically, the investigation first evaluates the behaviour of the coefficients ψ_1, λ_1 and the cost level $\hat{C}$ signalling replacement due to changes in the prevailing remain cumulative depreciation charge $\hat{X}$. Then we compare the effect of alternative periodic depreciation rates on $\hat{C}$ and contrast these with previous findings. Finally, we investigate the impact of volatility changes on the solution. The analysis is initially performed on the base case and then various parametric changes are introduced. The base case is specified in Table 2.

Table 2

C_0	X_0	K	N	D_N	τ	r	θ_C	σ_C
40	80	100	8	10	30%	20%	15%	20%

As the previously explained method has indicated, the solution is derived from a pre-specified value of the remaining cumulative depreciation charge $\hat{X}$. However, we express the behaviours in terms of the time spent since the previous re-investment $\hat{T}$. When the lifetime of the depreciation charge N and the initial remaining depreciation charge X_0 are set, then $\hat{X}$ determines $\hat{T}$ through $\hat{T} = N\left(1 - \dfrac{\hat{X}}{X_0}\right)$. Expressing the various behaviours in terms of $\hat{T}$ rather than $\hat{X}$ is preferable because it is more meaningful and enables the two regimes $\hat{T} \le N$ and $\hat{T} > N$ to be incorporated on a single graph.

The profiles for ψ_1 and λ_1, which are exhibited in Figure 7, clearly reveal the asymmetry at the depletion event for the remaining cumulative depreciation charge, $\hat{T} = N$. When $\hat{T} \le N$, both the profiles are declining functions of $\hat{T}$: λ_1 declines to the value zero and ψ_1 to 1.2846 at $\hat{T} = N$, and they then remain at these respective values for $\hat{T} > N$. When the remaining cumulative depreciation charge is completely depleted, $\hat{X}$ plays no further part in determining the optimal replacement policy. The asymmetry effect is also clearly visible at $\hat{T} = N$ in Figure 8, which exhibits the joint behaviour of the optimal values signalling replacement for the operating cost level, $\hat{C}$ and time $\hat{T}$. The profile relating $\hat{C}$ and $\hat{T}$ indicates the line of indifference between continuance with the incumbent asset and its replacement. Whenever the prevailing operating cost level for a certain time since the previous re-investment lies beneath the profile, the optimal decision is to continue with the incumbent asset but whenever the prevailing operating cost lies above the profile for a certain time, then the optimal decision is replacement. For $\hat{T} > N$, the line of indifference between continuance and replacement is independent of $\hat{T}$ since the remaining cumulative depreciation charge is completely depleted. In contrast, there exists a positive relationship between $\hat{C}$ and $\hat{T}$ when $\hat{T} \leq N$. The tolerance for replacing younger assets is less than for older assets and the degree of tolerance increases with asset age until it reaches a maximum at the depletion event for the remaining cumulative depreciation charge. Young machines that experience significantly high operating cost levels will be replaced.

The effects of altering the depreciation lifetime for the asset are displayed in Figure 9, which exhibits the various lines of indifference for $N = 20$, $N = 8$, $N = 4$ and $N = 0$. This figure also presents the lie of indifference for a zero depreciation amount, when $X_0 = D_N = 0$, which is the solution to the model formulated by Dobbs (2004) and represented by (25). When the depreciation lifetime tends to zero $N \to 0$, then the various quantities involving depreciation in (21) and (23) adopt the following values:

$$\lim_{N \to 0} \left\{ \frac{D_N \tau}{r} \left(1 - \exp\left(\frac{-r\hat{X}}{D_N} \right) \right) \right\} \to 0, \; \lim_{N \to 0} \tau\hat{X} \to 0$$

and by l'Hospital's rule:

$$\lim_{N \to 0} \left\{ \frac{D_N \tau}{r} \left(1 - \exp\left(\frac{-rX_0}{D_N} \right) \right) \right\} \to X_0 \tau .$$

As N decreases and the point of asymmetry shifts leftwards along the line of indifference, the horizontal component to its right the reflecting (23) declines in value while the slope of the component to its left reflecting (21) increases in value. As N approaches zero, the component to the left of the point of asymmetry becomes increasingly more insignificant and the horizontal component dominates. In contrast, as N becomes increasingly large, then:

$$\lim_{N\to\infty}\left\{\frac{D_N\tau}{r}\left(1-\exp\left(\frac{-rX_0}{D_N}\right)\right)\right\}\to 0$$

(24) and (25) are identical so the horizontal component to the right of the point of asymmetry tends the solution value proposed by Dobbs (2004). The advantages of the solution values yielded by Dobbs (2004) are its ease in calculation and the provision of an upper limit. However, the present formulation demonstrates that a more efficient upper limit is supplied by the horizontal component to the right of the point of asymmetry (24) and that the two solution methods share a similar degree of computational ease.

We now examine the effect of volatility changes on the solution. When $\sigma_C = 0$, the solutions for the base case to the deterministic model from (28), $\hat{T} = 5.6315$ and to the stochastic model from (17), (21) and (22), $\hat{X} = 23.6847$ and $\hat{C} = 93.0940$, are identical in line with the analytical proof given in the Appendix C. It can be shown numerically that positive changes in the volatility produce negative changes in both ψ_1 and λ_1 but $\hat{C}$ and $\hat{F} = F\left(\hat{C}, \hat{X}\right)$ are both increasing functions of σ. The profiles for $\hat{C}$ and $\hat{F}$ are exhibited in Figure 10 for $\hat{X} = 23.6847$; in evaluating $\hat{F}$ we ignore the revenue term, which explains its negative value. The behaviour of these profiles agree with the findings of previous work on real options analysis and replacement models.

6. Comparison of Declining Balance and Straight Line Methods

The simulation results for the replacement real option model under declining balance and straight line depreciation schedules presented in sections 3 and 5 respectively offer no guidance on the comparative merits of the two alternative schedules. The aim of this section is to compare the replacement policies for the two schedules under reasonably similar conditions in order to identify the circumstances favouring one schedule relative to the other and to discern whether either of the two alternatives can be classified as ideal. Creating similar conditions underpinning the simulation exercise first requires setting the variables common to both depreciation schedule to be identical. Secondly, we stipulate that the implied expected lifetime for the incumbent asset to be set to be equal for the two depreciation schedules. These two requirements imply that $D_0 = \theta_D K$ for the declining balance method and $D_C = K$ for the straight line method, with $1/\theta_D = N/2$.

Table 3
Base Case Data for Comparing the Effects
of the Two Depreciation Schedules

Common Data		
Replacement investment cost	K	100
Initial operating cost for a replica	C_0	10
Risk neutral operating cost drift rate	θ_C	4%
Operating cost volatility	σ_C	25%
Risk-free interest rate	r	7%
Relevant corporate tax rate	τ	30%
Declining Balance Method		
Initial depreciation charge	D_0	10
Depreciation declining balance rate	θ_D	10%
Straight Line Method		
Cumulative depreciation at replacement	X_0	100
Length of depreciation duration	N	20

In the presence of operating cost uncertainty, the comparison of the effects of the two depreciation schedules on the replacement policy is simulated using the data presented in Table 3. The data values for the common factors across the alternative depreciation schedules are set to be identical. The remaining parameters, which are distinctive due to the depreciation specification, are compelled to be comparable. The discriminatory boundaries for the replacement model under the two depreciation schedules against the age of the incumbent asset are presented in Figure 11.

The preferred depreciation schedule for the stochastic replacement model ought to universally encourage the accelerated replacement of the incumbent asset relative to its contender so that the productive process always experiences an incumbent asset that suffers the less input decay. This criterion implies that the preferred depreciation schedule ought to furnish the lower operating cost trigger level for all asset ages. Figure 11 reveals that there is no definitive winner. Although the declining balance schedule is preferred for newly installed assets, its position changes with asset age. From approximately 2 – 10 years, which is when the incumbent attains its expected life according to the depreciation schedules, the straight line schedule furnishes a lower operating cost trigger level and so represents the preferred method. The depreciation tax allowance under the declining balance schedule continues to decline with asset age. Over the critical range, the depreciation tax allowance for the straight line schedule becomes relatively more pronounced as the asset ages, but its residual depreciation tax allowance declines. The magnitude of the preference for the straight line method over this range is not significantly large with a proportional change not exceeding 1%. When the incumbent reaches its expected life according to the depreciation schedules, the preferred method of depreciation reverts to declining balance. From this age onwards, the residual depreciation tax allowance under the declining balance method becomes relatively more pronounced and this causes comparatively accelerated replacement owing to effects on the net replacement investment cost. Although there exists no universally ideal depreciation method, the declining balance schedule is to be preferred since the magnitude of the preference when it is second choice is only quite small.

7. Conclusion

In this paper, we analyse the replacement model for a productive asset that is subject to input decay, depreciation tax allowances and a fixed investment cost. Previous real option models on the stochastic replacement phenomenon have concentrated solely on single factors representations. Although the model proposed by Mauer & Ott (1995) involves three variables, these are condensed to a single variable by forcing depreciation and salvage value to be functions of the stochastic operating cost. Real option models specified for different contexts and involving more than a single factor either invoke the property of homogeneity of degree one where it is logically valid in order to reduce model dimensionality to a tractable level or resort to a purely numerical solution method. The property of homogeneity of degree one does not hold for the formulation under current study. Instead, we use an analytical approach to determining the levels triggering replacement for the two factors by specifying the form of the valuation function and deriving the trigger levels from the economic boundary conditions.

The quasi-analytical approach to determining the trigger levels has several comparative advantages. We demonstrate that from the result for the general replacement model, we can derive the special cases of a zero depreciation variable and for a zero operating cost volatility. These derived results are shown to be identical to the single factor stochastic replacement model under risk neutrality as proposed by Dobbs (2004) and the deterministic model under the dynamic programming formulation. Further, it is possible in principle to determine key indicators such as vega. The numerical results corroborate the findings of similar past works by establishing that both the operating cost trigger level and the valuation function increase for positive changes in the underlying volatility.

By analysing the replacement model under alternative depreciation schedules, it is possible to discern the preferred form of schedule that comparatively accelerates the replacement event. Although there exists no definitive victor, the schedule based on the declining balance method is preferred for most asset ages and when it is second choice, the difference in the operating cost trigger levels for the two methods is relatively slight. By permitting the depreciation schedule to adopt one of two forms, a time dependent variable is included in the formulation and the resulting valuation function depends on two distinct factors. This two factor model is investigated through the quasi-analytical approach which yields a set of simultaneous equations from which the trigger levels can be generated. This approach has the potential that it can be extended to analysing multi-factor real option models that involve a time dependent variable. This means that finite-lived assets with embedded options, such as those whose productive life is constrained by external obligations and natural resources, can now in principle be evaluated using this approach.

Appendix A: Deterministic Replacement Model

In this appendix, we examine the replacement models in a deterministic world where the depreciation charge is measured by (i) the declining balance and (ii) the straight line method. The notation used here employs the subscript indexed by the time variable t since the key variables are time dependent and the optimal solution is expressed in time units.

(i) Declining Balance Depreciation Charge

The present value V_T for an asset with a lifetime T measured in years is the discounted future after tax net cash flows at the annualised continuous risk-adjusted rate of μ:

$$V_T = \int_0^T \left\{ (1-\tau) P_0 - (1-\tau) C_0 e^{\alpha_C t} + \tau D_0 e^{-\alpha_D t} \right\} e^{-\mu t} \, dt . \tag{26}$$

The asset is financially viable for some definite lifetime so $\alpha_C < \mu$. By adapting the result by Lutz & Lutz (1951), the optimal replacement time for the asset is found from maximising the value of the infinite chain V_∞ with respect to T, where:

$$V_\infty = V_T + \left(V_\infty + \tau RD_T - K \right) e^{-\mu T}$$

where RD_T denotes the residual depreciation charge at replacement. Using the notation $\tilde{}$ to represent a variable's optimal value, the first order condition for V_∞ to attain a maximum is:

$$\left. \frac{dV_T}{dT} \right|_{T=\tilde{T}} = \left(\mu \left(V_\infty + \tau RD_{\tilde{T}} - K \right) - \tau \left. \frac{dRD}{dT} \right|_{T=\tilde{T}} \right) e^{-\mu \tilde{T}} = 0 . \tag{27}$$

The optimal solution depends on the specification for the residual depreciation charge. There are two alternatives. Under type A, the unused depreciation charge is granted as a single amount allowed against tax, then $RD_T = \frac{D_T}{\theta}$ and $\left. \frac{dRD}{dT} \right|_{T=T^*} = -D_{T^*}$. (27) simplifies to:

$$\frac{(1-\tau) C_{T^*}}{\mu} \left(1 + \frac{\alpha_C e^{-\mu T^*}}{\mu - \alpha_C} \right) - \tau D_{T^*} \left(\frac{e^{-\mu T^*}}{\mu + \theta} - \frac{1}{\theta} \right) = \frac{(1-\tau) C_0}{\mu - \alpha_C} - \frac{\tau D_0}{\mu + \theta} + K . \tag{28}$$

Under type B, residual depreciation allowance against tax is the present value of the unused depreciation charges discounted at μ, then $RD_T = \frac{D_T}{\mu+\theta}$ and $\left.\frac{dRD}{dT}\right|_{T=T^*} = -\frac{D_T}{\mu+\theta}$.

(27) simplifies to:

$$\frac{(1-\tau)C_{T*}}{\mu}\left(1+\frac{\alpha_C e^{-\mu T*}}{\mu-\alpha_C}\right)=\frac{(1-\tau)C_0}{\mu-\alpha_C}-\frac{\tau D_0}{\mu+\theta}+K. \qquad (29)$$

The optimal replacement time increases from type A to B because of the enhanced tax credit on replacement.

(ii) Straight Line Depreciation Charge

Under the straight line method, the cumulative depreciation charge for capital allowance purposes, denoted by D_C, is equally apportioned over the asset's presumed lifetime of N. The periodic depreciation charge, D_N, is given by $D_N = D_C / N$ if the time point of interest is not greater than N and $D_N = 0$ if otherwise.

When $T \le N$, the present value for the asset becomes:

$$V_T = \int_0^T \left\{(1-\tau)P_0 - (1-\tau)C_0 e^{\alpha_C t} + \tau D_N\right\} e^{-\mu t}\, dt.$$

The first order optimality condition is given by (27). Then under type A, $RD_T = D_N(N-T)$ with $\frac{dRD_T}{dT} = -D_N$, and the optimal solution simplifies to:

$$\frac{(1-\tau)C_{\tilde{T}}}{\mu}\left(1+\frac{\alpha e^{-\mu\tilde{T}}}{\mu-\alpha}\right)+\frac{\tau D_N}{\mu}\left(1-e^{-\mu\tilde{T}}\right)=\frac{(1-\tau)C_0}{\mu-\alpha}-\tau D_N\left(N-\tilde{T}\right)+K \qquad (30)$$

When $T > N$, the present value for the asset becomes:

$$V_T = \int_0^T \left\{(1-\tau)P_0 - (1-\tau)C_0 e^{\alpha t}\right\} e^{-\mu t}\, dt + \int_0^N \tau D_N e^{-\mu t} dt.$$

The value of the infinite chain becomes:

$$V_\infty = V_T + \left(V_\infty - K\right) e^{-\mu T}.$$

And the first order optimality condition becomes:

$$\left.\frac{dV_T}{dT}\right|_{T=T^*} = \mu\left(V_\infty - K\right)e^{-\mu T^*}$$

It is straightforward to derive the optimal solution:

$$\frac{\left(1-\tau\right)C_{\tilde{T}}}{\mu}\left(1+\frac{\alpha e^{-\mu\tilde{T}}}{\mu-\alpha}\right)+\frac{\tau D_N}{\mu}\left(1-e^{-\mu N}\right)=\frac{\left(1-\tau\right)C_0}{\mu-\alpha}+K\,. \qquad (31)$$

The differences between (31) and (28) are the use of N instead of $\tilde{T}$ on the left hand side and the omission of the residual depreciation charge allowance against tax on the right hand side.

Appendix B: Contingent Claims Analysis

Under a contingent claims formulation, a portfolio is constructed of one long unit of the project F and ϖ short units of the operating cost C. When this portfolio is held over the short time interval $\left(t, t+dt\right)$, it accrues a capital appreciation and cash flow gain from its various constituents. These are shown in the following table:

	F	ϖC
Capital appreciation	dF	ϖdC
Cash flow gain	$\left(\left(P_0 - C\right)\left(1-\tau\right)+D\tau\right)dt$	$\varpi\phi C dt$

The coefficient ϕ represents the dividend yield for the traded security twinned with C.

Operating costs and the depreciation charge follow a geometric Brownian process (1) and a geometric deterministic process (2) respectively. The overall gain for the portfolio over the short time interval $\left(t, t+dt\right)$ is:

$$\left(dF - \varpi dC\right)+\left(\left(P-C\right)\left(1-\tau\right)+D\tau-\varpi\phi C\right)dt\,.$$

By invoking Ito's lemma and setting $\varpi = \frac{\partial F}{\partial C}$ to eliminate terms in dZ, the overall gain for the portfolio becomes:

$$\left(\tfrac{1}{2}\sigma^2 C^2 \frac{\partial^2 F}{\partial C^2} - \phi C \frac{\partial F}{\partial C} - \alpha_D D \frac{\partial F}{\partial D} + (P - C)(1-\tau) + D\tau \right) dt .$$

Since this portfolio enjoys a risk-free gain, the return on the portfolio value depends on the risk-free rate r so:

$$r\left(F - C\frac{\partial F}{\partial C} \right) dt = \left(\tfrac{1}{2}\sigma^2 C^2 \frac{\partial^2 F}{\partial C^2} - \phi C \frac{\partial F}{\partial C} - \alpha_D D \frac{\partial F}{\partial D} + (P - C)(1-\tau) + D\tau \right) dt.$$

Re-arranging, the risk neutral valuation relationship for the project F becomes:

$$\tfrac{1}{2}\sigma^2 C^2 \frac{\partial^2 F}{\partial C^2} + (r - \phi) C \frac{\partial F}{\partial C} - \alpha_D D \frac{\partial F}{\partial D} + (P - C)(1-\tau) + D\tau - rF = 0. \quad (32)$$

When the deterministic process for the depreciation charge is arithmetic (14), the risk neutral valuation relationship becomes:

$$\tfrac{1}{2}\sigma^2 C^2 \frac{\partial^2 F}{\partial C^2} + (r - \phi) C \frac{\partial F}{\partial C} - \alpha_D \frac{\partial F}{\partial D} + (P - C)(1-\tau) + D\tau - rF = 0. \quad (33)$$

Except for the coefficient change, (32) and (33) are respectively identical to (3) and (16).

Appendix C

Zero variance

The stochastic model is recast within a dynamic programming framework by setting $\alpha_D = \theta_D$, $\alpha_C = \theta_C$ and $\mu = r$. When $\sigma_C = 0$, (6) simplifies to $\alpha_C \eta_1 - \alpha_D \beta_1 = \mu$. Further,

$$\left(\frac{D_0}{\hat{D}} \right)^{\beta_1} \left(\frac{C_0}{\hat{C}} \right)^{\eta_1} = e^{-\mu \hat{T}} .$$

By making these substitutions in (10), it is straightforward to demonstrate that the stochastic model simplifies to (28).

Mauer and Ott

Mauer & Ott (1995) treat depreciation as a function of cost and set the depreciation tax shield over t to $t + dt$ equal to:

$$\tau D_0 \left(\frac{C}{C_0} \right)^{-\frac{\theta}{z}}$$

where $z = \alpha_C - \frac{1}{2}\sigma_C^2$. Ignoring the salvage price on disposal, their valuation relationship is represented by the partial differential equation:

$$\tfrac{1}{2}\sigma_C^2 C^2 \frac{\partial^2 F_{MO}}{\partial C^2} + \alpha_C C \frac{\partial F_{MO}}{\partial C} + P_0(1-\tau) - C(1-\tau) + k_1 \tau C^{\delta} - \mu F_{MO} = 0,$$

where $k_1 = D_0 C_0^{-\delta}$ and $\delta = -\dfrac{\theta}{z}$. The solution to this partial differential equation is:

$$F_{MOD} = A_{MOD} C^{\eta_{MO}} + \frac{P_0(1-\tau)}{\mu} - \frac{C_0(1-\tau)}{\mu - \alpha_C} + \frac{k_1 C^{\delta}}{\Lambda}, \tag{34}$$

where:

$$\eta_{MO} = \left(\tfrac{1}{2} - \frac{\alpha_C}{\sigma_C^2}\right) + \sqrt{\left(\tfrac{1}{2} - \frac{\alpha_C}{\sigma_C^2}\right)^2 + \frac{2\mu}{\sigma_C^2}},$$

and:

$$\Lambda = \mu - \alpha_C \delta - \tfrac{1}{2}\sigma_C^2 \delta(\delta - 1).$$

For the sake of comparison, we have ignored their reflecting barrier condition. Cancelling out the revenue term on either side of the equation, the value matching condition becomes:

$$\begin{aligned} A_{MOD}\hat{C}_{MOD}^{\eta_{MO}} - \frac{\hat{C}_{MOD}(1-\tau)}{\mu-\alpha_C} + \frac{k_1 \tau \hat{C}_{MOD}^{\delta}}{\Lambda} \\ = A_{MOD} C_0^{\eta_{MO}} - \frac{C_0(1-\tau)}{\mu-\alpha_C} + \frac{k_1 \tau C_0^{\delta}}{\Lambda} + \frac{k_1 \tau \hat{C}_{MOD}^{\delta}}{\theta} - K, \end{aligned} \tag{35}$$

where $\hat{C}_{MOD}$ represents the optimal cost trigger level under their formulation. Using the smooth pasting condition:

$$\eta_{MO} A_{MOD} \hat{C}_{MOD}^{\eta_{MO}-1} - \frac{(1-\tau)}{\mu-\alpha_C} + \frac{\delta k_1 \tau \hat{C}_{MOD}^{\delta-1}}{\Lambda} = \frac{\delta k_1 \tau \hat{C}_{MOD}^{\delta-1}}{\theta}.$$

(35) simplifies to:

$$\begin{aligned} \frac{C_0(1-\tau)}{\eta(\mu-\alpha_C)} - \frac{k_1 \hat{C}_0^{\delta}\tau}{\Lambda} + K = \frac{\hat{C}_{MOD}(1-\tau)}{\eta_{MO}(\mu-\alpha_C)}\left(\eta_{MO} - 1 + \left(\frac{C_0}{\hat{C}_{MOD}}\right)^{\eta_{MO}}\right) \\ + \frac{k_1 \hat{C}_{MOD}^{\delta}\tau}{\eta_{MO}}\left(\eta_{MO} - \delta + \left(\frac{C_0}{\hat{C}_{MOD}}\right)^{\eta_{MO}}\right)\left(\frac{1}{\theta} - \frac{1}{\Lambda}\right), \end{aligned} \tag{36}$$

from which the optimal cost trigger level can be implicitly evaluated.

Now, we focus on salvage price in the formulation of Mauer & Ott (1995) and ignore the depreciation charge. The authors assume that salvage price is an inverse function of the cost: $S = \frac{k_2}{C}$, where k_2 is an exogenously specified constant. The value matching condition becomes:

$$\overline{A}_{MOS}\hat{C}_{MOS}^{\eta_{MO}} - \frac{\hat{C}_{MOS}(1-\tau)}{\mu-\alpha_C} = \overline{A}_{MOS}C_0^{\eta_{MO}} - \frac{C_0(1-\tau)}{\mu-\alpha_C} + \frac{k_2(1-\tau)}{\hat{C}_{MOS}} - K, \tag{37}$$

where $\hat{C}_{MOS}$ denotes the cost trigger level. Using the smooth pasting condition:

$$\eta_{MO}\overline{A}_{MOS}\hat{C}_{MOS}^{\eta_{MO}-1} - \frac{(1-\tau)}{\mu-\alpha_C} = -\frac{k_2(1-\tau)}{\hat{C}_{MOS}^2}$$

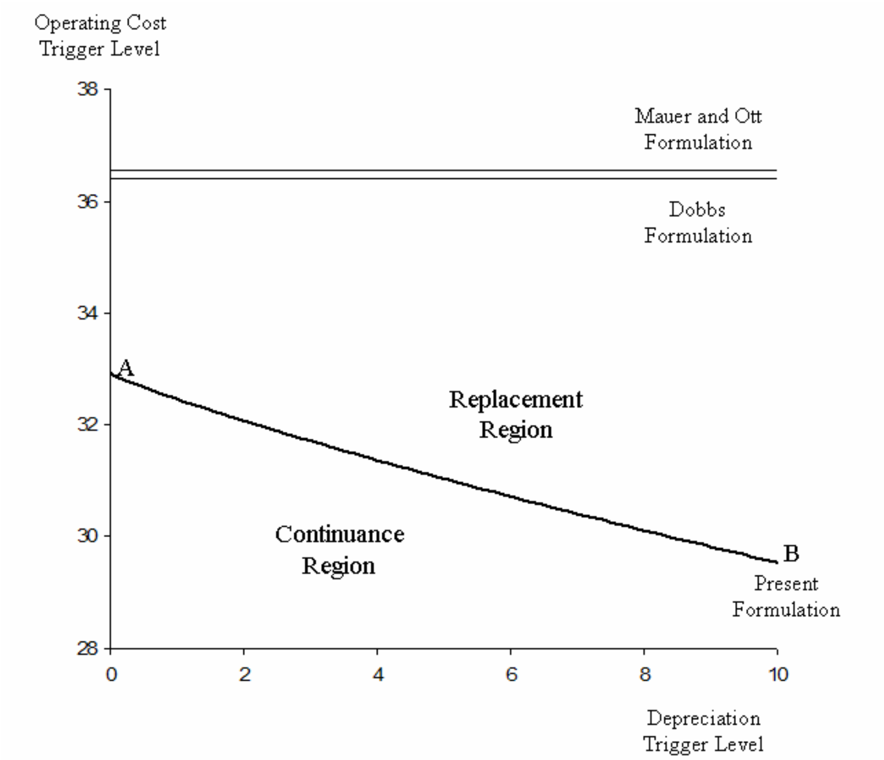
(37) simplifies to:

$$\frac{\hat{C}_{MOS}(1-\tau)}{\eta_{MO}(\mu-\alpha_C)}\left(\eta_{MO}-1+\left(\frac{C_0}{\hat{C}_{MOS}}\right)^{\eta_{MO}}\right) + \frac{k_2(1-\tau)}{\eta_{MO}\hat{C}_{MOS}}\left(\eta_{MO}+1-\left(\frac{C_0}{\hat{C}_{MOS}}\right)^{\eta_{MO}}\right) = \frac{C_0(1-\tau)}{\eta(\mu-\alpha_C)} + K, \tag{38}$$

$$\frac{\hat{C}_{MOS}(1-\tau)}{\eta_{MO}(\mu-\alpha_C)}\left(\eta_{MO}-1+\left(\frac{C_0}{\hat{C}_{MOS}}\right)^{\eta_{MO}}\right) + \frac{k_2(1-\tau)}{\eta_{MO}\hat{C}_{MOS}}\left(\eta_{MO}+1-\left(\frac{C_0}{\hat{C}_{MOS}}\right)^{\eta_{MO}}\right) = \frac{C_0(1-\tau)}{\eta(\mu-\alpha_C)} + K,$$

from which the optimal cost trigger level can be implicitly evaluated.

Figure 1: Variations between the Operating Cost and Depreciation Trigger Levels



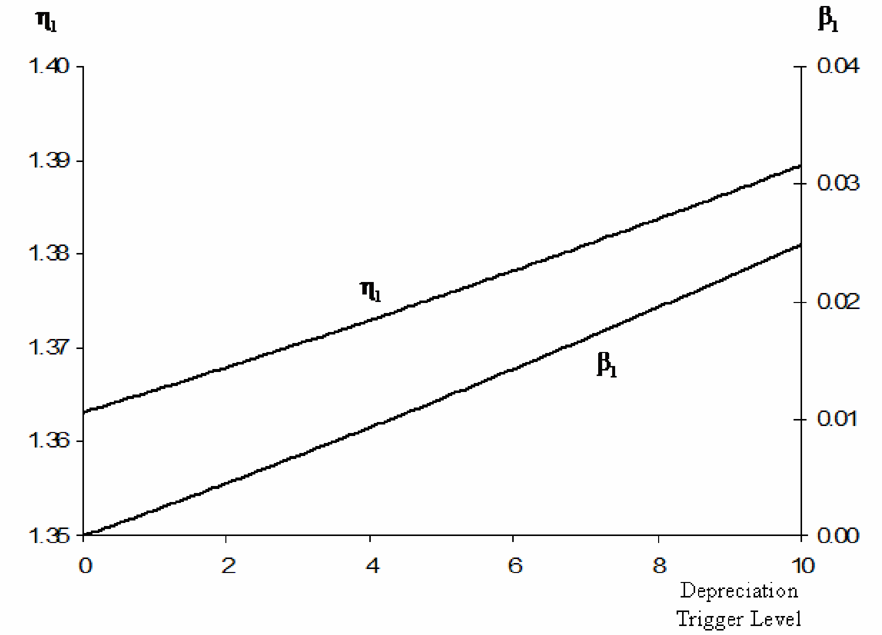
This figure is based on calculations using the following information:

C_0	D_0	K	θ_C	θ_D	r	σ_C	τ
10	10	100	4%	10%	7%	25%	30%

The operating cost trigger levels for the Dobbs (2004) model and the Mauer & Ott (1995) model are determined from **Error! Reference source not found.** and (36) respectively; the operating cost trigger level for both these formulations is independent of the depreciation trigger level. The profile of the operating cost trigger and the depreciation trigger levels for the current formulation is determined from **Error! Reference source not found.** and **Error! Reference source not found.** for the range of $\hat{D}$ from zero to 10. Typical pairs of trigger levels are presented in the following table:

$\hat{D}$	0.0	5.0	10.0
$\hat{C}$	32.92	31.04	29.54

Figure 2: Profile of the Parameters β_1 and η_1 for Variations in the Depreciation Trigger Level



Typical values for the parameters β_1, η_1 and the depreciation trigger level $\hat{D}$ are shown in the following table:

$\hat{D}$	β_1	η_1
0.0	0.0000	1.363
5.0	0.0117	1.376
10.0	0.0249	1.389

Figure 3:

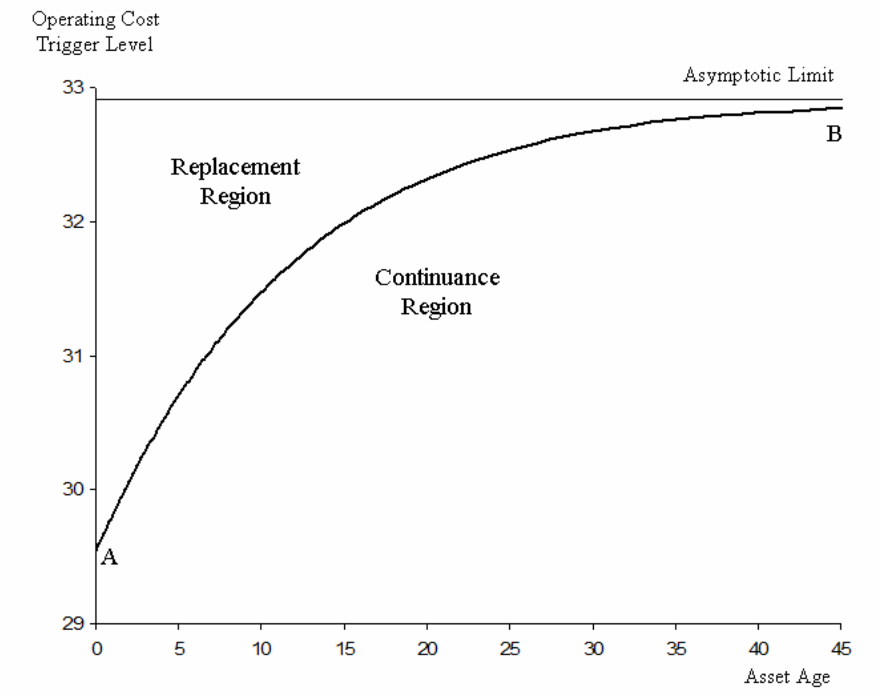

Operating Cost
Trigger Level
Asymptotic Limit
33
32
31
30
29
B
Replacement
Region
Continuance
Region
A
0
5
10
15
20
25
30
35
40
45
Asset Age

Figure 7:

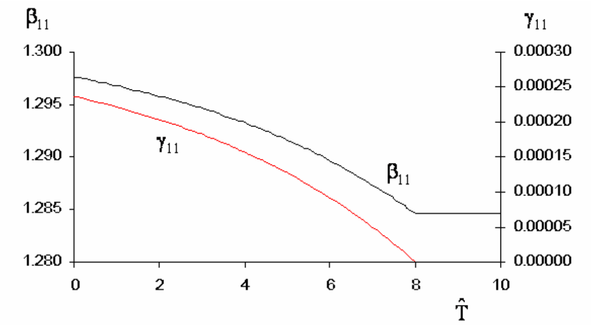


Figure 8:

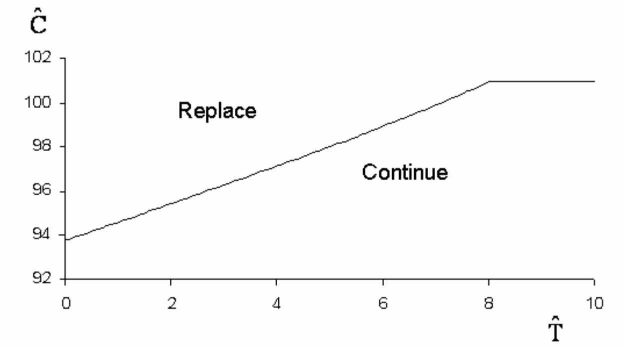


Figure 9

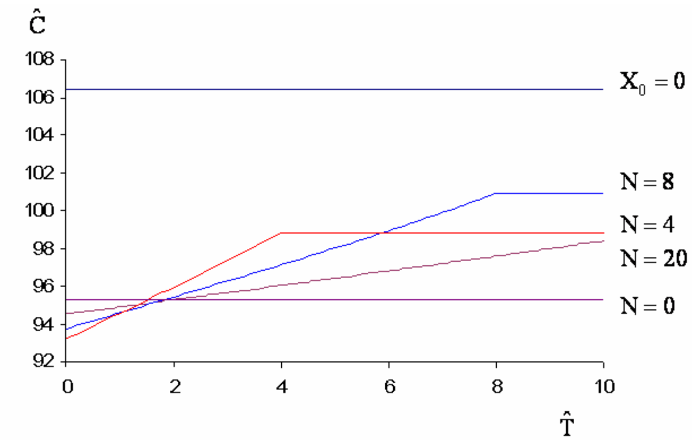


Figure 10

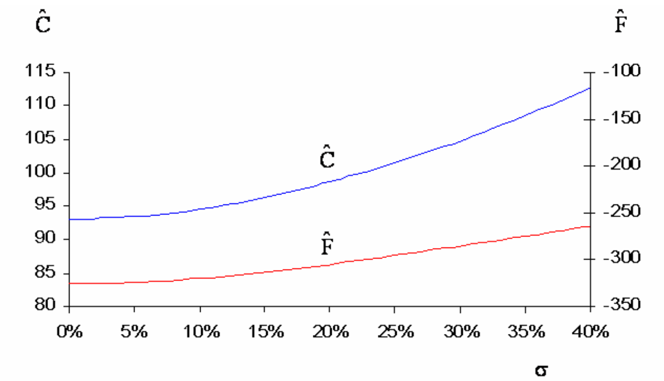
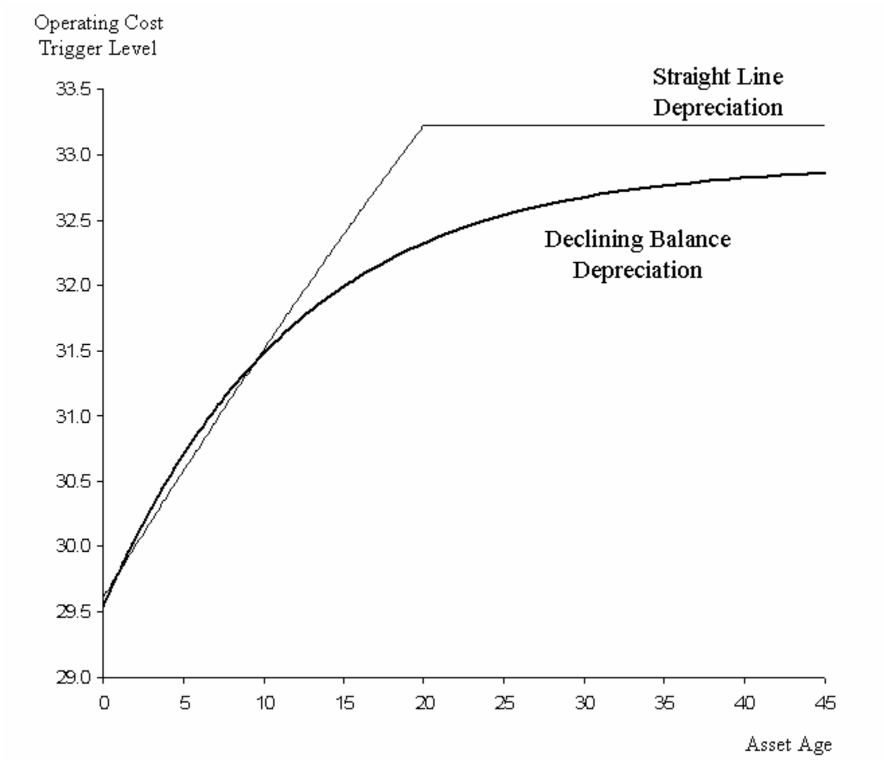


Figure 11: Comparison of Operating Cost Trigger Level versus Asset Age for Depreciation Schedules based on Declining Balance and Straight Line Method



References

Adkins, R., & Paxson, D. (2006). Optionality in asset renewals, Real Options Conference. New York.

Bellman, R. E. (1955). Equipment replacement policy. Journal of the Society for Industrial and Applied Mathematics, 3(3), 133-136.

Brennan, M. J., & Schwartz, E. S. (1985). Evaluating natural resource investments. Journal of Business, 58(2), 135-157.

Constantinides, G. M. (1978). Market risk adjustment in project valuation. Journal of Finance, 33(2), 603-616.

Dixit, A. K., & Pindyck, R. S. (1994). Investment under Uncertainty. Princeton (N.J.): Princeton University Press.

Dobbs, I. M. (2004). Replacement investment: Optimal economic life under uncertainty. Journal of Business Finance & Accounting, 31(5-6), 729-757.

Feldstein, M. S., & Rothschild, M. (1974). Towards an economic theory of replacement investment. Econometrica, 42(3), 393-424.

Lutz, F., & Lutz, V. (1951). The Theory of Investment of the Firm. Princeton (N.J.): Princeton University Press.

Malchow-Møller, N., & Thorsen, B. J. (2005). Repeated real options: Optimal investment behaviour and a good rule of thumb. Journal of Economic Dynamics & Control, 29(6), 1025-1041.

Malchow-Møller, N., & Thorsen, B. J. (2006). Corrigendum to "Repeated real options: Optimal investment behaviour and a good rule of thumb". Journal of Economic Dynamics & Control, 30(5), 899.

Margrabe, W. (1978). The value of an option to exchange one asset for another. Journal of Finance, 33(1), 177-186.

Mason, S. P., & Merton, R. C. (1985). The role of contingent claims analysis in corporate finance. In E. I. Altman & M. G. Subrahmanyam (Eds.), Recent Advances in Corporate Finance. Homewood, Illinois: Richard D. Irwin.

Mauer, D. C., & Ott, S. H. (1995). Investment under uncertainty: The case of replacement investment decisions. Journal of Financial and Quantitative Analysis, 30(4), 581-605.

McDonald, R. L., & Siegel, D. R. (1985). Investment and the valuation of firms when there is an option to shut down. International Economic Review, 26(2), 331-349.

McDonald, R. L., & Siegel, D. R. (1986). The value of waiting to invest. The Quarterly Journal of Economics, 101(4), 707-728.

Myers, S. C., & Turnball, S. M. (1976). Capital budgeting and the capital asset pricing model: Good news and bad news. Journal of Finance, 32(2), 321-333.

Rust, J. (1987). Optimal replacement of GMC bus engines: An empirical model of Harold Zurcher. Econometrica, 55(5), 999-1033.

Samuelson, P. A. (1965). Rational theory of warrant pricing. Industrial Management Review, 6(Spring), 41-50.

Shimko, D. C. (1992). Finance in Continuous Time: A Primer. Miami: Kolb Publishing Company.

Sick, G. (1989). Capital Budgeting with Real Options: Salomon Brothers Center for the Study of Financial Institutions, New York University

Tourinho, O. A. F. (1979). The valuation of reserves of natural resources: an option pricing approach: University of California, Berkeley.

Trigeorgis, L. (1996). Real Options: Managerial Flexibility and Strategy in Resource Allocation. Cambridge (Mass.): MIT Press.

Williams, J. T. (1991). Real estate development as an option. Journal of Real Estate Finance and Economics, 4, 191-208.

Williams, J. T. (1997). Redevelopment of real assets. Real Estate Economics, 25(3), 387-407.

Ye, M.-H. (1990). Optimal replacement policy with stochastic maintenance and operation costs. European Journal of Operational Research, 44, 84-94.